where erf($S$) is the error function defined by

$$\operatorname{erf}(S) = \frac{2}{\pi^{1/2}} \int_0^S e^{-x^2}\, dx \tag{13}$$

and

$$S = U/(2RT)^{1/2} = (\gamma/2)^{1/2} M$$

From the definition of the stress tensor, one obtains for the diagonal terms

$$P_{ii} = 3p$$

also

$$p_{ij} = P_{ij} - p\delta_{ij} \tag{14}$$

and hence

$$p_{ij} = 0 \text{ (i.e., } p_{xx} + p_{yy} + p_{zz} = 0)$$

From the foregoing relationship, if one substitutes $-(p_{xx} + p_{yy})$ for $p_{zz}$, both $p_{zz}$ and $p_{yy}$ terms drop out.

One should note that these results are confined to the case of a monatomic gas, since there still exists a question as to the nature of the distribution function analogous to that given by Eq. (5) but applying to a diatomic gas. For adiabatic wall conditions $Q = 0$. If $T_w$ is denoted by $T_{aw}$, the resulting equilibrium sphere temperature can be written as

$$\frac{T_{aw}}{T} = H_0 + \frac{p_{xx}}{p} H_{pxx} + \frac{q_x}{pU} H_{qx}$$

where

$$H_0 = \frac{\operatorname{erf}(S)(S^4 + 3S^2 + \frac{3}{4}) + [S \exp(-S^2)/\pi^{1/2}]\,(S^2 + \frac{5}{2})}{\operatorname{erf}(S)(2S^2 + 1) + (2S/\pi^{1/2})e^{-S^2}}$$

$$H_{pxx} = \frac{\operatorname{erf}(S)(3S^4 - 3S^2 + \frac{1}{4}) + [S \exp(-S^2)/\pi^{1/2}](3S^2 - \frac{1}{2}) + [\operatorname{erf}(S)(2S^2 + 2) - 4S \exp(-S^2)/\pi^{1/2}(S^2 + 1)]H_0}{4[\operatorname{erf}(S)(2S^2 + 1) + (2Se^{-S^2}/\pi^{1/2})]S^2} \tag{15}$$

$$H_{qx} = \frac{4\{\operatorname{erf}(S)(S^2 + \frac{3}{2}) - [S \exp(-S^2)/\pi^{1/2}](5S^2 + 3) + [\frac{1}{4}\operatorname{erf}(S) - S \exp(-S^2)/2\pi^{1/2}]H_0\}}{5[\operatorname{erf}(S)(2S^2 + 1) + (2S/\pi^{1/2})e^{-S^2}]}$$

These partial equilibrium temperature ratios are plotted in Fig. 2, where they indicate that for $S \gg 1$ the effects due to viscous stresses and heat flux become relatively unimportant.

For the heat transfer case, one defines a Stanton number $St$ for sphere:

$$St = Nu/RePr = [\alpha(\gamma - 1)/\gamma]B(S)$$

where

$$B(S) = \left[\operatorname{erf}(S)(2S^2 + 1) + \frac{2S}{\pi^{1/2}} \exp(-S^2)\right] \frac{1}{S^2} + \frac{p_{xx}}{4p}\left[\operatorname{erf}(S)(2S^2 + 2) - \frac{4S}{\pi^{1/2}} \exp(-S^2)(S^2 + 1)\right] - \frac{q_x S^2}{5pU}\left[\frac{1}{4}\operatorname{erf}(S) - \frac{S}{\pi^{1/2}} \exp(-S^2)\right] \tag{16}$$

$$\frac{1}{\alpha} St = St_0 + \frac{p_{xx}}{p} St p_{xx} - \frac{q_x}{pU} St_{qx} \tag{17}$$

where

$$\frac{1}{\alpha} St_0 = \frac{2}{5}\left[\operatorname{erf}(S)(2S^2 + 1) + \frac{2S}{\pi^{1/2}} \exp(-S^2)\right] \cdot \frac{1}{S^2}$$

$$\frac{1}{\alpha} St_{pxx} = \frac{1}{10S^4}\left[\operatorname{erf}(S)(2S^2 + 2) - \frac{4S}{\pi^{1/2}} \exp(-S^2)(S^2 + 1)\right]$$

$$\frac{1}{\alpha} Stq_x = \left[\frac{\operatorname{erf}(S)}{4} - \frac{S}{2\pi^{1/2}} \exp(-S^2)\right] \frac{2}{25S^2} \tag{18}$$

These partial Stanton numbers are plotted in Fig. 3.

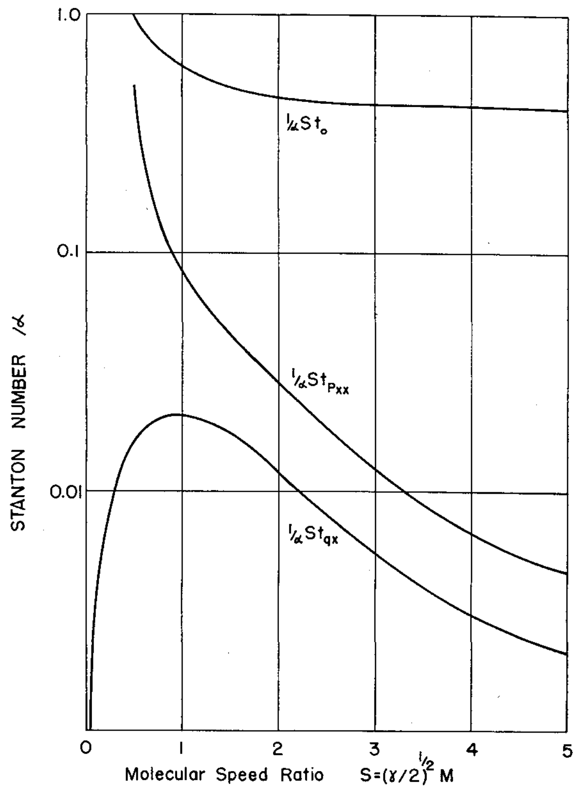


**Fig. 3 Partial Stanton numbers for nonuniform free molecule flow past a sphere.**


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# Acoustic Probe for Hypersonic Air Data Sensing

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### Nomenclature

$u$ = gas velocity
$\gamma$ = isentropic expansion exponent
$P$ = pressure
$\rho$ = density
$a$ = sonic velocity

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$T$ = temperature
$C_p$ = specific heat at constant pressure
$R$ = universal gas constant/molecular weight
$E_d$ = dissociation energy per unit mass
$\tau$ = two-way transit time of an acoustic wave between probe and shock front
$\Delta_0$ = standoff distance of shock wave on the axis
$\epsilon$ = density ratio across the shock wave
$r_s$ = distance from center to shoulder of flat-nosed body
$\nu$ = frequency of acoustic wave
$\delta_\nu$ = Doppler shift in $\nu$

## Introduction

THE problem of determining the flow parameters in the hypersonic flow regime by means of probes carried on an aerospace vehicle has been complicated by two factors: 1) inability to describe analytically the subsonic flow regime downstream of the shock because of uncertainty about the chemistry of dissociation, ionization, and recombination; and 2) short lifetime of the probe resulting from its intense heating by the hypersonic stream. One solution to this problem has been proposed[1] and is currently under investigation.[2] One aspect of the approach of Ref. 1 also was discussed in Ref. 3. In this paper an alternative approach is proposed. The objective is to select parameters for measurement which may be expected to be less affected by the uncertain aerothermochemical processes than conventional pitot tube measurements, while being more sensitive to changes in the hypersonic flow parameters. To be sure, the difficulties just mentioned will still have to be overcome. However, some progress is being made on these problems, and their solution may be within the state of the art of the next few years.

## Measurement Parameters

It is well known that the standoff distance of a detached shock wave becomes increasingly less sensitive to changes in freestream Mach number, at fixed freestream pressure and density, as the Mach number becomes very large. Thus, shock wave standoff distance, by itself, would not be a very good parameter to use in computing hypersonic Mach numbers or velocities. On the other hand, stagnation pressure and stagnation temperature should become increasingly sensitive to Mach number as the Mach number is increased. Now, stagnation pressure should be fairly easy to measure, but the same cannot be said for stagnation temperature.

To gain some insight into the kinds of measurements which may prove useful, it is instructive to consider the following simple analysis. The conservation equations across a normal shock wave are

$$(U_1^2/2) + C_{p1}T_1 = (U_2^2/2) + C_{p2}T_2 + E_d \quad (1)$$

$$P_1 + \rho_1 U_1^2 = P_2 + \rho_2 U_2^2 \quad (2)$$

$$\rho_1 U_1 = \rho_2 U_2 \quad (3)$$

For a perfect gas, $C_p$ is given by

$$C_p = \gamma R/(\gamma - 1) \quad (4)$$

Assume that the air upstream of the shock wave is at a temperature of 300°R, is flowing at a velocity of 30,000 fps, and is undissociated. Assume that the downstream air is flowing at a velocity of 3160 fps and is completely dissociated. These conditions correspond roughly to those which would be encountered at a standard altitude of 240,000 ft. Assume air to be a simple diatomic gas with $\gamma = 1.4$ and $R = 1715$ ft²/sec²-°R. Complete dissociation will change these values to $\gamma = 1.67$ and $R = 3430$ ft²/sec²-°R. For the undissociated and dissociated cases, respectively, Eq. (4) gives $(C_p)_{un} =$ 6010 ft-lb/°R-slug, $(C_p)_d = 8560$ ft-lb/°R-slug. Neglecting ionization and taking the energy of dissociation to be $3.06 \times 10^8$ ft-lb/slug yields for $T_2$, from Eq. (1),

$$T_2 = \frac{1}{8560}\left[6010 \times 300 + \frac{(30{,}000)^2}{2} - \frac{(3160)^2}{2} - 3.06 \times 10^8\right] = 15{,}750°\text{R}$$

It is clear from the foregoing expression that the terms $C_{p1}T_1$ and $U_2^2/2$ may be neglected. Were it not for the term $E_d$, the upstream velocity would then be given by

$$U_1 \approx [2/(\gamma - 1)]^{1/2} a_2 \quad (5)$$

that is, the downstream sonic velocity in the hypersonic regime would be almost directly proportional to the upstream gas velocity.

To include real gas effects, the approach of Ref. 4 was used, in which the following reactions are considered:

$$O_2 \rightleftarrows 2O$$

$$N_2 \rightleftarrows 2N$$

$$O \rightleftarrows O^+ + e^-$$

$$N \rightleftarrows N^+ + e^-$$

Reference 4 obtains a closed-form solution for the equilibrium thermodynamic properties of air which provides data within 5% of the exact data of Ref. 5. Figure 1 presents some results from a Northrop Norair computer program,[6, 7] giving the downstream sonic velocity as a function of freestream velocity for several altitudes. Equation (5) also is plotted for comparison. The waves in the results are a consequence of the chemical changes occurring across the shock wave. The wave crest occurring at approximately 14,000 fps denotes completion of oxygen dissociation, whereas the crest at 30,000 fps denotes completion of nitrogen dissociation. In spite of the waviness, the sonic velocity remains a monotonically increasing function of freestream velocity. Therefore, it would be a suitable variable to detect in order to obtain a measure of freestream velocity, provided that some variable related to altitude were simultaneously detected.

One possibility for measuring the average sonic velocity in the subsonic region is to generate an acoustic wave at the probe, reflect it from the shock front, and detect its return to the probe. If the standoff distance of the shock wave is determined by measurement or calculation, a measurement of the time $\tau$ between emission of the acoustic wave and its return to the probe should yield the acoustic velocity.

The theory relating the standoff distance of a detached shock wave to the aerophysical parameters is rather cumbersome even when simplifying assumptions as constant density or constant stream-tube area are introduced. Nevertheless, some fairly simple formulas can be extracted to give a qualitative picture of the situation. For an axially symmetric flat-nosed cylinder of radius $r_s$, the standoff distance at the axis of symmetry $\Delta_0$ is given by[8]

$$\Delta_0 = (3)^{3/4}\epsilon^{1/2} r_s \quad (6)$$

where $\epsilon$ is the density ratio across the shock $\rho_1/\rho_2$. An expression for $\epsilon$, for the case of a perfect gas, may be obtained from the conservation relations across the shock, that is,

$$\epsilon = \frac{\gamma - 1}{\gamma + 1}\left(1 + \frac{2}{\gamma - 1} M_1^{-2}\right) \quad (7)$$

In the limit of very large initial Mach number, this becomes

$$\epsilon_{\text{lim}} = (\gamma - 1)/(\gamma + 1) \quad (8)$$

All monatomic gases have $\epsilon_{\text{lim}} = \frac{1}{4}$, and diatomic gases have $\epsilon_{\text{lim}} = \frac{1}{6}$. In air at elevated temperatures, $\epsilon_{\text{lim}}$ may drop to a value of the order of 0.07 or less because of the effects of dissociation and ionization. For air, including real gas effects, the curves of $\epsilon$ vs $u_1$, with altitude as a parameter, are quite wavy. Despite this, the ratio $\Delta_0/r_s$ does not vary by more than a factor of 1.55 between the extreme values of $\epsilon$ likely to be encountered during hypersonic re-entry, as may be seen from Table 1.

**Table 1 Shock standoff distance**

| Altitude, ft | Freestream speed, fps | $\epsilon$ | $\Delta_0/r_s$ |
|---|---|---|---|
| 300,000 | 27,000 | 0.0476 | 0.492 |
| 300,000 | 13,000 | 0.0834 | 0.659 |
| 150,000 | 28,000 | 0.0616 | 0.565 |
| 150,000 | 10,000 | 0.111 | 0.761 |

If we were to consider $\Delta_0/r_s$ constant at its geometrical mean value of 0.611, we would err at the extremes by the ratio

$$0.761/0.611 = 0.611/0.492 = 1.24$$

Thus, we would make an error of 24% in the sonic velocity at the extremes if we were able to measure the transit time of the acoustic wave accurately. This error can be reduced by making use of another measurement, as will be shown below.

Another measurement that can be made easily, beside transit time of an acoustic wave, is the stagnation pressure in the subsonic region $P_{2_s}$. In the hypersonic regime, the momentum equation (2) reduces to the approximate relationship

$$P_{2_s} \approx \rho_1 u_1^2 \tag{9}$$

The accuracy of Eq. (9) is indicated in Table 2, where the same example values are used as in Table 1.

**Table 2 Exact vs approximate stagnation pressure**

| Altitude, ft | Freestream speed, fps | Freestream density, slugs/ft³ | $\rho_1 u_1^2$, psf | $P_{2_s}$, psf |
|---|---|---|---|---|
| 300,000 | 27,000 | $6.065 \times 10^{-9}$ | 4.41 | 4.25 |
| 300,000 | 13,000 | $6.065 \times 10^{-9}$ | 1.03 | 1.00 |
| 150,000 | 28,000 | $3.564 \times 10^{-6}$ | 2780 | 2900 |
| 150,000 | 10,000 | $3.564 \times 10^{-6}$ | 356 | 380 |

The maximum error for these values is 6.5%. Thus, Eq. (9) probably may be considered sufficiently accurate for computing purposes, although it is easy to correct it if it should be necessary.

Yet another measurement is required in order to define the flow completely. A possibility is to use the Doppler shift $\delta\nu$ of the frequency $\nu$ of the acoustic wave, produced by the gas velocity downstream of the shock wave. Taking account of the shift before and after reflection from the shock wave, one can easily show that the total shift is

$$\delta\nu = \frac{(u_2/a_2)^2}{1 - (u_2/a_2)^2}\,\nu \tag{10}$$

To illustrate the magnitudes involved, consider a probe having a shoulder radius $r_s$ equal to 0.2 ft. Using the conditions of a previous example, take $u_1 = 30{,}000$ fps at an altitude of 240,000 ft and $u_2 = 3160$ fps. From Table 1 it is estimated that $\Delta_0/r_s$ is about 0.5. Then $\Delta_0 = 0.1$ ft.

Assume, for the present, that the shock front reflects like a closed tube. Although this may not seem justifiable on the basis of the static pressure ratio across the shock wave, it does seem so if one considers the stagnation pressure ratio instead. It is proposed that the actual reflection conditions of an acoustic wave at a shock front be determined by a theoretical study.

On the basis of the foregoing assumption, the fundamental resonant wavelength, around which it is proposed to operate, is given by $\lambda = 4(\Delta_0) = 4 \times 0.1 = 0.4$ ft. Then the frequency is given by $\lambda = 8950/0.4 = 22{,}400$ cps. Introducing these values into Eq. (10) yields $\delta\nu = 0.1 \times 22{,}400 = 2240$ cps. These values were all obtained on the basis of one-dimensional theory, and the Doppler shift for axially symmetric three-dimensional flow into a stagnation point will be less than that just computed. However, the computed value was large enough so that, even if the actual value were only one-third as big, it would still be easily measurable.

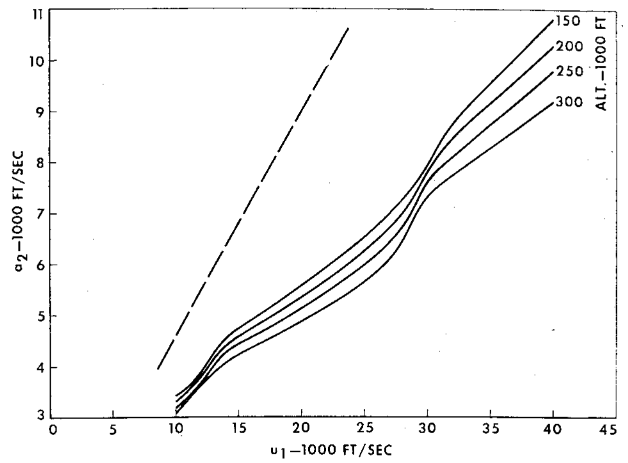


**Fig. 1 Downstream sonic velocity vs upstream gas velocity.**

The procedure for calculating hypersonic aerophysical data from measured quantities in the proposed system may now be summarized as follows. The measured quantities will be $\tau$, $\delta\nu$, and $P_{2_s}$. The calculated quantities will be $a_2$, $u_1$, and $\rho_1$. Three formulas from which these quantities may be calculated are

$$a_2 = \left(\frac{\delta\nu}{\nu + \delta\nu}\right)^{1/2} \frac{(3)^{3/2}(r_s)^2}{u_1\tau} \tag{11}$$

obtained by combining Eqs. (3, 6, and 10) and writing $\Delta_0 = a_2\tau$,

$$u_1 = 4.01(a_2 - 878)\ln(2.72 \times 10^{-3}/\rho_1)^{0.187} \tag{12}$$

obtained as a rough empirical fit of Fig. 1; and Eq. (9). In the worst case, Eq. (12) leads to an underestimation of $u_1$ by about 10%. Although a more accurate empirical relation could be obtained at the cost of more complexity, it probably is not warranted unless the effects of heat radiation and non-equilibrium flow are included.

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